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Cylindrically symmetric scalar waves in general relativity

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Abstract : In this note, exact solutions of Einstein equations with scalar waves are obtained for the most general cylindrically symmetric space-time which reduce to essentially static forms. The asymptotic behaviour of the null geodesic near the curvature singularity of a solution is discussed. The other solution is found to have no finite curvature singularity

Keywords : Einstein equations with scalar waves, cylindrically symmetric space-time, exact solutions

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General relativity couples gravity with all fields. The study of the exact solutions of gravity coupled to other fields is important to understand clearly the physical and mathematical structures of space-times. For many reasons, the study of Einstein equations in the presence of scalar fields has been an object of special attention and various aspects of the problem have been investigated by Brahmchary [1], Bergmann and Leipnik [2], Buchdahl [3], Janis *et al* [4], Penny [5], Gautreau [6] and others. Most of the authors have taken up the problems of interacting gravitational and scalar fields with and without the rest-mass term. Several physically acceptable scalar-tensor theories of gravitation have been proposed and are widely studied by many workers. Scalar-tensor theories of gravitation provide the most natural generalisations of general relativity and thus provide a convenient set of representations for the observational limits on possible deviations from general relativity.

The most general spherically symmetric static solution of Einstein equations coupled with mass-less scalar field was found by Wyman [7]. Since then, some authors investigated its global properties and a few interesting results were found. Roberts [8] has discussed the applications of spherically symmetric solutions of the mass-less scalar Einstein equations to cosmic censorship and has given a non-static solution to the field equations. He has also constructed the Vaidya form of Wyman solution obeying the reasonable energy conditions. Li and Liang [9] have presented the static general solution with plane symmetric scalar fields and have shown that the singularity in the plane symmetric case is not influenced essentially by the introduction of the scalar field. Li [10] has presented the general plane symmetric metric yielded by a scalar wave and concluded that the metric is either static or spatially homogeneous. He has shown that the Taub Theorem [11] can be generalised to space-time with a scalar wave. Shri Ram and Singh [12] have derived an exact non-static scalar wave solution for the cylindrically symmetric Marder [13] metric which give Taub solution [11] and Li solution [10] in special cases.

In this note, we consider Einstein equations with scalar wave for the most general cylindrically symmetric metric recently discussed by Banerjee *et al* [14] in the investigation of exact gravitational fields due to static and nonstatic cosmic strings arising due to the breaking of a global U (1) symmetry. The field equations are completely integrated and two exact solutions are then presented which reduce to essentially static form under coordinate transformations. We also discuss the asymptotic behaviour of the null geodesic near the singularity of one of the solutions. The other solution has no finite singularity.

Field equations :

The general cylindrically symmetric line element can be written as

$$ds^2 = e^{2(K-U)}(-dt^2 + dr^2) + e^{2U}dz^2 + W^2 e^{-2U} d\theta^2, \tag{1}$$

where all of K, U and W are functions of r and t [14]. Setting $x^1 = r$, $x^2 = z$, $x^3 = \theta$ and $x^4 = t$, the non-vanishing components of the Ricci tensor are R_{11}, R_{22}, R_{33}, R_{44} and R_{14}.

The energy-momentum tensor for a massless scalar field is

$$T_{\alpha\beta} = \phi_{,\alpha}\,\phi_{,\beta} - \frac{1}{2} g_{\alpha\beta}\phi_{,\gamma}\,\phi^{,\gamma}, \tag{2}$$

where the scalar field ϕ is the solution of Klein-Gordon equation :

$$g^{\alpha\beta}\phi_{;\alpha\beta} = 0. \tag{3}$$

A comma and a semicolon denote ordinary and covariant derivative respectively.

The Einstein equations are

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = 8\pi T_{\alpha\beta}. \tag{4}$$

On contraction, the field equations (4) can be written in the form

$$R_{\alpha\beta} = 8\pi\phi_{,\alpha}\,\phi_{,\beta}. \tag{5}$$

Because of the symmetry in the metric (1), ϕ is function of r and t. For the line-element (1), the Einstein equations (5) give the following set of equations :

$$-\frac{W_{11}}{W} - K_{11} + K_{44} + U_{11} - U_{44} + \frac{U_1 W_1}{W} + \frac{K_1 W_1}{W} - \frac{U_4 W_4}{W} + \frac{K_4 W_4}{W} - 2U_1^2 = 8\pi\phi_1^2, \tag{6}$$

$$-\frac{W_{11}}{W} + \frac{W_{44}}{W} + U_{11} - U_{44} + \frac{U_1 W_1}{W} - \frac{U_4 W_4}{W} = 0, \tag{7}$$

$$-U_{11} + U_{44} - \frac{U_1 W_1}{W} + \frac{U_4 W_4}{W} = 0, \tag{8}$$

$$-\frac{W_{44}}{W} + K_{11} - K_{44} - U_{11} - U_{44} + \frac{K_1 W_1}{W} - \frac{U_1 W_1}{W} + \frac{K_4 W_4}{W} + \frac{U_4 W_4}{W} - 2U_4^2 = 8\pi\phi_4^2, \tag{9}$$

$$-\frac{W_{14}}{W} + \frac{W_1 W_4}{W} + \frac{K_1 W_4}{W} - 2U_1 U_4 = 8\pi\phi_1\phi_4. \tag{10}$$

The Klein-Gordon equation (3) leads to

$$\phi_{11} - \phi_{44} + \frac{W_1 \phi_1}{W} - \frac{W_4 \phi_4}{W} = 0. \tag{11}$$

A linear combination of eqs (6–10) yields

$$W_{44} - W_{11} = 0, \tag{12}$$

$$-U_{11} + U_{44} - \frac{U_1 W_1}{W} + \frac{U_4 W_4}{W} = 0, \tag{13}$$

$$-\frac{W_{11}}{W} - \frac{W_{44}}{W} + \frac{2K_1 W_1}{W} + \frac{2K_4 W_4}{W} - 2U_1^2 - 2U_4^2 = 8\pi\left[\phi_1^2 + \phi_4^2\right], \tag{14}$$

$$K_{44} - K_{11} + U_{11} - U_{44} - U_1^2 + U_4^2 + \frac{U_1 W_1}{W} - \frac{U_4 W_4}{W} = 4\pi\left[\phi_1^2 - \phi_4^2\right], \tag{15}$$

and

$$-\frac{W_{14}}{W} + \frac{W_1 K_4}{W} + \frac{W_4 K_1}{W} - 2U_1 U_4 = 8\pi\phi_1\phi_4. \tag{16}$$

Here suffixes 1 and 4 are differentiation with respect to r and t respectively.

Solutions of the field equations :

The general solution of (12) is

$$W = w_1(\xi) + w_2(\eta), \quad \text{where } \xi = t + r,\ \eta = t - r. \tag{17}$$

Let the scalar field ϕ be the solution of wave equation, which is referred to as a scalar wave.

$$\phi = h_1(\xi) + h_2(\eta). \tag{18}$$

Using (17) and (18) in (11) we obtain

$$\frac{dh_1}{d\xi} \Big/ \frac{dw_1}{d\xi} = -\frac{dh_2}{d\eta} \Big/ \frac{dw_2}{d\eta} = a,$$

where a is an arbitrary constant. In view of these equations, the scalar field ϕ becomes

$$\phi = a\{w_1(\xi) - w_2(\eta)\} + b, \tag{19}$$

b being another arbitrary constant. From eqs. (12) and (13), we can write

$$U = \epsilon \log W, \quad \epsilon = \pm 1 \tag{20}$$

Using (17), (18) and (19) in eq. (15), we obtain

$$\frac{\partial^2 K}{\partial\xi\partial\eta} = 4\pi a^2 w_1'(\xi) w_2'(\eta) - \frac{w_1'(\xi) w_2'(\eta)}{[w_1(\xi) + w_2(\eta)]^2}. \tag{21}$$

A dash denotes ordinary derivative. Eq. (21) has the general solution

$$K = 4\pi a^2 w_1(\xi) w_2(\eta) + \log\{w_1(\xi) + w_2(\eta)\} + g_1(\xi) + g_2(\eta), \tag{22}$$

where $g_1(\xi)$ and $g_2(\eta)$ are arbitrary functions. Substituting K from (22) into (14) and (16), we obtain

$$w_1'(\xi) g_1'(\xi) + w_2'(\eta) g_2'(\eta) = \frac{1}{2}[w_1''(\xi) + w_2''(\eta)]$$
$$+ 4\pi a^2 \left[\{w_1'(\xi)\}^2 w_1(\xi) + \{w_2'(\eta)\}^2 w_2(\eta)\right]. \tag{23}$$

$$w_1'(\xi) g_1'(\xi) - w_2'(\eta) g_2'(\eta) = \frac{1}{2}[w_1''(\xi) - w_2''(\eta)]$$
$$+ 4\pi a^2 \left[\{w_1'(\xi)\}^2 w_1(\xi) - \{w_2'(\eta)\}^2 w_2(\eta)\right]. \tag{24}$$

which are equivalent to

$$g_1(\xi) = \frac{1}{2}\log w_1'(\xi) + 2\pi a^2 \{w_1(\xi)\}^2 + \frac{1}{2}\log c_1 \tag{25}$$

and

$$g_2(\eta) = \frac{1}{2}\log w_2'(\eta) - 2\pi a^2 \{w_2(\eta)\}^2 + \frac{1}{2}\log c_2. \tag{26}$$

Equations (25) and (26) can be written as

$$e^{2g_1} = c_1 |w_1'| e^{4\pi a^2 w_1^2} = \pm c_1 w_1' e^{4\pi a^2 w_1^2}, \tag{27}$$

$$e^{2g_2} = c_2 |w_2'| e^{4\pi a^2 w_2^2} = \pm c_2 w_2' e^{4\pi a^2 w_2^2}. \tag{28}$$

Case I : when $\epsilon = -1$

From eqs. (20), (22), (27) and (28), we obtain

$$e^{2K-2U} = \pm c^2 \left[w_1(\xi) + w_2(\eta)\right]^4 w_1'(\xi) w_2'(\eta)\, e^{4\pi a^2 [w_1(\xi)+w_2(\eta)]^2}. \quad (29)$$

where we take the negative sign if $w_1' w_2' < 0$ and $c^2 = c_1 c_2$ is a constant.

The metric of the solution becomes

$$ds^2 = \pm c^2 \left[w_1(\xi) + w_2(\eta)\right]^4 w_1'(\xi) w_2'(\eta) e^{4\pi a^2 [w_1(\xi)+w_2(\eta)]^2} (d\xi d\eta) + \left[w_1(\xi) + w_2(\eta)\right]^{-2} dz^2 + \left[w_1(\xi) + w_2(\eta)\right]^4 d\theta^2. \quad (30)$$

Using scale transformation,

$$w_1(\xi) = R+T \quad \text{and} \quad w_2(\eta) = R-T,$$

the cylindrically symmetric line element (1) yielded by a scalar wave can be written in the form

$$ds^2 = c^2 R^4 e^{4\pi a^2 R^2} (dR^2 - dT^2) + R^{-2} dz^2 + R^4 d\theta^2. \quad (31)$$

The scalar curvature of space-time (31) has the value $32\pi a^2 / R^4 e^{4\pi a^2 R^2}$ which tends to infinity as $R \to 0$. Thus $R = 0$ is a scalar curvature singularity. Investigating the asymptotic behaviour of the null geodesic, it is found that the null geodesics approaching $R = 0$ in T–R plane are incomplete.

Case II : when $\epsilon = +1$

From eqs. (20), (22), (27) and (28), we obtain

$$e^{2K-2U} = \pm c^2 w_1'(\xi) w_2'(\eta) e^{4\pi a^2 [w_1(\xi)+w_2(\eta)]^2}, \quad (32)$$

where we take the negative sign if $w_1' w_2' < 0$ and $c^2 = c_1 c_2$ is a constant.

The metric of the solution becomes

$$ds^2 = c^2 w_1'(\xi) w_2'(\eta) e^{4\pi a^2 [w_1(\xi)+w_2(\eta)]^2} d\xi d\eta + \left[w_1(\xi) + w_2(\eta)\right]^2 dz^2 + d\theta^2. \quad (33)$$

Using scale transformation,

$$w_1(\xi) = R+T \quad \text{and} \quad w_2(\eta) = R-T,$$

the cylindrically symmetric line element (1) yielded by a scalar wave can be written as

$$ds^2 = c^2 e^{4\pi a^2 R^2} (dR^2 - dT^2) + R^2 dz^2 + d\theta^2. \quad (34)$$

The scalar curvature of space-time (34) is $32\pi a^2 / e^{4\pi a^2 R^2}$ which shows that the metric (34) has no finite singularity.

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